Period 3, Jan 13, 2025

c) $\lim_{x \to 0^+} x(\ln x)$

(Tip: think about the graph of $\ln x$ at $x = 0$)

$\lim_{x \to 0} \ln x = -\infty \qquad \lim_{x \to 0} x = 0$

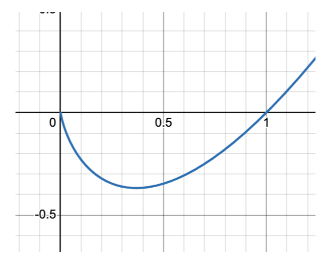
$0 \cdot -\infty = ?$

$x \ln x = \frac{\ln x}{\frac{1}{x}} = \frac{\ln x}{x^{-1}}$

$\lim_{x \to 0^+} \frac{\ln x}{x^{-1}} = \frac{\infty}{\infty} = \emptyset$

$\lim_{x \to 0^+} \frac{\ln x}{x^{-1}} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{-1x^{-2}} = \lim_{x \to 0^+} \frac{1}{x} \cdot -x^2 = -x = -0 = 0$

$y = x \ln x$



$g'(0)$

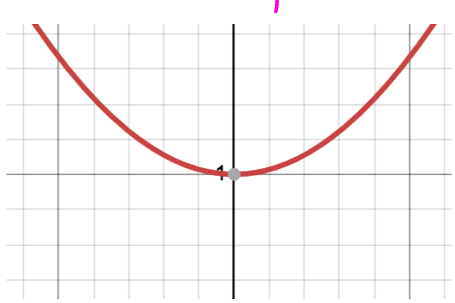
8. Use the limit to find the horizontal asymptotes of $g(t) = \frac{3t^2+2}{\sqrt{t^2+4}}$

$g(x) = \frac{3x^2+2}{(x^2+4)^{\frac{1}{2}}}$

$g'(x) = \frac{6x(x^2+4)^{\frac{1}{2}} - (3x^2+2)(\cancel{\tfrac{1}{2}}(x^2+4)^{-\frac{1}{2}} \cdot \cancel{2}x)}{((x^2+4)^{\frac{1}{2}})^2} = \frac{\frac{6x\sqrt{x^2+4}}{1 \cdot \sqrt{x^2+4}} - \frac{x(3x^2+2)}{\sqrt{x^2+4}}}{x^2+4}$

$\sqrt{x^2+4}$

$g'(x) = \frac{\frac{6x(x^2+4) - x(3x^2+2)}{\sqrt{x^2+4}}}{(x^2+4)} = \frac{6x^3+24x-3x^3-2x}{(x^2+4)\sqrt{x^2+4}} = 0$



$3x^3+22x=0$

$x(3x^2+22)=0$

$x=0$

1. If $g(x) = 2x^3$, find the equation of the tangent line to the point (1,2) , and use this linear approximations to estimate $g(1.3)$.

$g'(x) = 6x^2$

$g'(1) = 6(1)^2 = 6$ = slope

Point (1,2)

$y = 6x + b$

$2 = 6(1) + b$

$-4 = b$

$y = 6x - 4$

$y = 6(1.3) - 4 = 7.8 - 4 = 3.8$

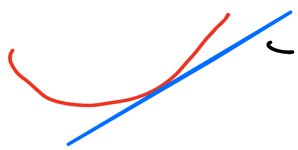
b) Is your approximation for $g(1.3)$ and overestimate or an underestimate? Give a reason for your answer.

$g'(x) = 6x^2$

$g''(x)$

$g''(x) = 12x$

$g''(1.3) = 12(1.3) = 15.6$ = positive concave up



← Tangent Line is below The curve

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2. If $f(x) = x^3 - 2x + 3$,

$F'(x) = 3x^2 - 2$

$F'(2) = 3(2)^2 - 2 = 10 = m$

a) Find the linearization $L(x)$ of $f(x)$ centered at $x = 2$.

$F(2) = 2^3 - 2(2) + 3 = 8 - 4 + 3 = 7$

(2,7)

$y = 10x + b$

$7 = 10(2) + b$

$-13 = b$

$L(x) = 10x - 13$

b) Use $L(x)$ to approximate $f(2.1)$

$10(2.1) - 13 = 21 - 13 = 8$

c) Determine if the approximation is an underestimate or an overestimate. Justify your answer.

$F''(x) = 6x$

$x = 2.1$

$F''(2.1) = 6(2.1)$ = positive (concave up)

under

d) *Use a calculator* to determine the accuracy of the approximation you found in part b.

$F(2.1) = (2.1)^3 - 2(2.1) + 3$

$L(2.1) = 8$

4. Find the differential for $y = \cos^2 x$ $\Rightarrow y = u^2$

$u = \cos x$

$\frac{du}{dx} = -\sin x$

$\frac{dy}{du} = 2u$

$\frac{du}{dx} \cdot \frac{dy}{du} = -\sin x \cdot 2u$

$dx \cdot \frac{dy}{dx} = -2\sin x \cos x \cdot dx$

$dy = -\underbrace{2\sin x \cos x}_{\sin 2x} dx$

$dy = -\sin 2x \, dx$

6. Use differentials to estimate the change in the volume of a sphere ($V = \frac{4}{3}\pi r^3$) when the radius changes from 10 cm to 10.05 cm.

$dr = 0.05$

$r = 10$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = \frac{4}{3}\pi \cdot 3r^2 = 4\pi r^2$

$dV = 4\pi r^2 dr$

$dV = 4 \cdot \pi \cdot 10^2 \cdot (0.05)$

5. Find dy and evaluate dy given value of x and dx.

$y = x^2 \ln x$, $x = 1$. $dx = 0.01$ $\Rightarrow dy = (2(1) \cdot \ln 1 + 1)(0.01) = (2 \cdot 0 + 1)(0.01)$

(0.01)

$\frac{dy}{dx} = 2x \cdot \ln x + x^2 \cdot \frac{1}{x} = 2x\ln x + x$

$dy = (2x\ln x + x) \cdot dx$

4. Given $m(t) = -2\sin(2t)$; $[-\pi, \pi]$, find the open intervals where the function is concave up and concave down. Justify your answer.

down

$\underbrace{}_{m''(T)=+}$ $\underbrace{}_{m''(T)=-}$

$y=-2\sin 2x \Rightarrow y=-2\sin u \Rightarrow \frac{dy}{dx}=-4\cos 2x$

$u=2x$ $\quad \frac{dy}{du}=-2\cos u$

$\frac{du}{dx}=2$

Same Process

$u=2x$

$\frac{d^2y}{dx^2}=+8\sin 2x$

$\cancel{\frac{du}{dx}}\cdot\frac{dy}{\cancel{du}}=\frac{dy}{dx}$

$2\cdot -2\cos 2x$

$0=m''(T)=8\sin 2T=8(2\sin T\cos T)$ $\quad [-\pi,\pi]$

$\sin T=0$ $\qquad \cos T=0$

$\sin -\pi=0$ $\qquad \cos -\frac{\pi}{2}=0$

$\sin \pi=0$ $\qquad \cos \frac{\pi}{2}=0$

$\sin 0=0$

$-\cdot-=+$ $\quad -\cdot+=-$ $\quad +\cdot+=+$ $\quad +\cdot-=-$

$+++++$ $\quad ----$ $\quad +++++$ $\quad ---$

Tests $\quad -\pi \quad -\frac{3\pi}{4} \quad -\frac{\pi}{2} \quad -\frac{\pi}{4} \quad 0 \quad \frac{\pi}{4} \quad \frac{\pi}{2} \quad \frac{3\pi}{4} \quad \pi$

Concave up $\quad$ Concave Down $\quad$ Concave up $\quad$ Concave Down

$\sin -\frac{3\pi}{4}=-\frac{\sqrt{2}}{2}$ $\qquad \cos -\frac{3\pi}{4}=-\frac{\sqrt{2}}{2}$

$\sin -\frac{\pi}{4}=-\frac{\sqrt{2}}{2}$ $\qquad \cos -\frac{\pi}{4}=\frac{\sqrt{2}}{2}$

$\sin \frac{\pi}{4}=\frac{\sqrt{2}}{2}$ $\qquad \cos \frac{\pi}{4}=\frac{\sqrt{2}}{2}$

$\sin \frac{3\pi}{4}=\frac{\sqrt{2}}{2}$ $\qquad \cos \frac{3\pi}{4}=-\frac{\sqrt{2}}{2}$

Example 1: Use Optimization techniques to find two numbers whose sum is 20 and whose product is as large as possible. What is the largest product?

a and b

$a \cdot b = y$

$10 \cdot 10 = 100$

$9 \cdot 11 = 99$

$a + b = 20$

$a = \boxed{20-b} = 20 - 10 = 10$

$\Rightarrow (20-b) \cdot b = y$

$20b - b^2 = y$

To Find max/mins $\Rightarrow \frac{dy}{dx} = 0$ or $\emptyset$

$20 - 2b = \frac{dy}{dx} \Rightarrow 0 - 2 = \frac{dy}{dx}$ = concave Down (max)

$20 - 2b = 0$

$20 = 2b$

$10 = b$

Example 2: You have 40 feet of fence to enclose a rectangular garden along the side of a lake. What is the maximum area that you can enclose?

$\underbrace{2x}_{-2x} + y = \underbrace{40}_{-2x}$

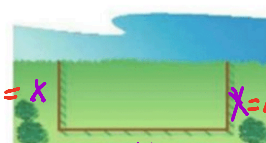
$y = \boxed{40 - 2x}$

$40 - 2(10)$

$40 - 20$

$y = 20$

Area $= xy$



$10 = x$ $\quad$ $x = 10$

$y = 20$

$A = x(40 - 2x)$

$A = 40x - 2x^2 \Rightarrow \frac{dA}{dx} = 40 - 4x$

Max/min

$\frac{dA}{dx} = 0$

$0 = 40 - 4x$

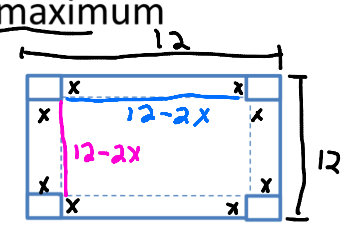
$4x = 40$

$x = 10$

$\frac{d^2A}{dx^2} = -4$ concave Down

max

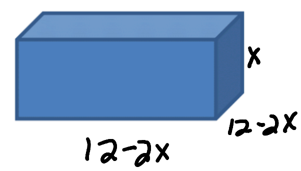
Example 3: An open-top box is to be made by cutting congruent squares of sides $\boldsymbol{x}$ from the corners of a 12- by 12-cm sheet of tin and bending up the sides. How large should the squares be to make the box hold as much as possible? What is the resulting maximum volume?



$$V = (12-2x)(12-2x)\cdot x$$

$$V = (144 - 48x + 4x^2)x$$

$$V = 144x - 48x^2 + 4x^3$$



$$\frac{dV}{dx} = 144 - 96x + 12x^2 = 12(x^2 - 8x + 12)$$

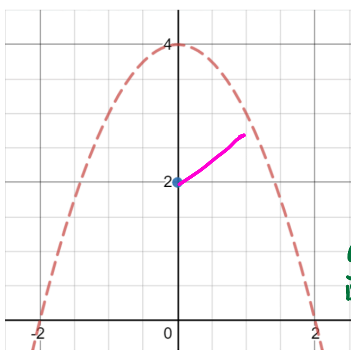
$$= 12(x-6)(x-2) = 0$$

$$x = 6 \text{ or } x = 2$$

$$\frac{d^2V}{dx^2} = -96 + 24x$$

$-96+24(6)$	$-96+24(2)$
$-96+144$ = Positive	$-96+48$ = negative
Concave Down	-48 Concave Down
Min	Max

Example 4: Which points of the graph of $y = 4 - x^2$ are closest to the point (0,2)?



$d = \sqrt{(x-0)^2 + (4-x^2-2)^2}$

$d^2 = x^2 + 4 - 4x^2 + x^4$

$d^2 = x^4 - 3x^2 - 4$

$2d \frac{dd}{dx} = 4x^3 - 6x = 2x(2x^2 - 3)$

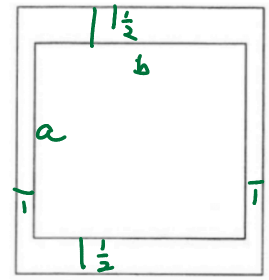
distance is positive

$= 0$

$x = 0$ or $x = \sqrt{\frac{3}{2}}$

min

Example 5: A rectangular page is to contain 24 square inches of print. The margins at the top and bottom of the page are to be 1 ½ inches, and the margins on the left and right are to be 1 inch. What should be the dimensions of the page be so that the least amount of paper is used?



$a \cdot b = 24 \Rightarrow a = \frac{24}{b}$

dimensions

$a+3 \qquad b+2$

$\frac{24}{b}+3 \qquad b+2$

$(a+3)(b+2)$ = amount of paper used

$\left(\frac{24}{b}+3\right)(b+2) = A$

$24 + 48b^{-1} + 3b + 6 = A$

$0 - 48b^{-2} + 3 = \frac{dA}{db}$

$-\frac{48}{b^2} + 3 = 0$

$\cancel{b^2} \cdot \frac{48}{\cancel{b^2}} = 3 \cdot b^2$

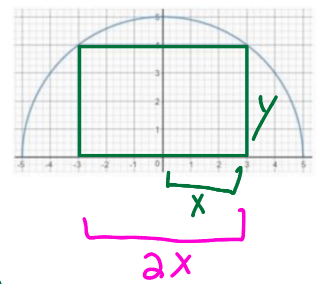
$\frac{48}{3} = \frac{3b^2}{3}$

$16 = b^2$

$4 = b$

$a = \frac{24}{b} = \frac{24}{4} = 6$

Example 6: A rectangle is bounded by the x-axis and the semicircle $y = \sqrt{25 - x^2}$. Find the dimensions of the rectangle that would maximize the area of the rectangle.



$A = 2x \cdot y$

$A = 2x(25 - x^2)^{\frac{1}{2}}$

$\frac{dA}{dx} = 2(25-x^2)^{\frac{1}{2}} + 2x\left(\cancel{\frac{1}{2}}(25-x^2)^{-\frac{1}{2}} \cdot -\cancel{2}x\right)$

$\frac{2\sqrt{25-x^2}\sqrt{25-x^2}}{1\sqrt{25-x^2}} - \frac{2x^2}{\sqrt{25-x^2}} = \frac{2(25-x^2)-2x^2}{\sqrt{25-x^2}}$

$x \neq 5$ or -5 $\frac{dA}{dx} = 0$

$50 - 2x^2 - 2x^2 = 0$

$50 = 4x^2$

$\frac{50}{4} = x^2$

$\sqrt{\frac{50}{4}} = x$

$\frac{5\sqrt{2}}{2} = x$